Flow and heat transfer due to natural convection in granular materials

M. Massoudi*, T.X. Phuoc

U.S. Department of Energy, Federal Energy Technology Center, P.O. Box 10940; MS: 84-340, Pittsburgh, PA 15236, USA

Received 12 October 1997; received in revised form 3 April 1998

Abstract

The present study considers an assembly of spherical particles, densely packed, between two vertical flat plates which are at different temperatures. The flow due to such a temperature difference is investigated. The governing equations for the flow of granular materials, taking into account the natural convection, are derived using a continuum model. For a fully developed flow of these materials, the equations reduce to a system of coupled, non-linear ordinary differential equations. The non-dimensional forms of the equations are integrated numerically. The results are then presented for volume fraction, dimensionless velocity, and temperature profiles as functions of various dimensionless parameters representing the following physical meaning: R_1 is the ratio of the pressure force to the force developed in the material due to distribution of the void. R_2 is the ratio of the viscous force to the gravity force and R_3 is the product of the Prandtl number and the Eckert number. © 1998 Elsevier Science Ltd. All rights reserved.

Keywords: Heat transfer; Natural convection; Continuum mechanics; Granular materials; Boundary value problems; Constitutive relations

1. Introduction

Interest in the behavior of granular materials has been primarily motivated by design problems in the bulk handling of grain, sand and gravel, powders, and other particulate solids. In processing minerals, in many chemical processes, and preventing or protecting against natural phenomena such as avalanches or debris flows, it is essential to understand the factors governing the packing and flow of powders and bulk solids [1]. Early studies of the flow of bulk solids were mainly concerned with the engineering and structural design of bins and silos. The inaccuracies of these theories, especially for the dynamic conditions of loading or emptying, occasionally resulted in the failure of the bin or silo. Furthermore, flowing granular materials can be considered as the limiting case of two-phase flow at high solid concentration and high solid-to-fluid density ratios [2]. In many applications, especially the cases of dense phase granular flows, such as fluidized beds, slurry transport, pneumatic and hydrolic transports, etc., one may have to use multiphase flow theories to

*Corresponding author. Tel.: 001 412 892 4975; fax: 001 412 892 6204; e-mail: massoudi@fetc.doe.gov.

accurately predict the flow and behavior of each phase [3, 4].

In general, the behavior of granular materials is very complex, and its understanding requires the merging of ideas from various fields of mechanics such as fluid mechanics, plasticity theory, rheology, and kinetic theory. Any theory attempting to describe the behavior of flowing granular materials should embody several features and some unique characteristics. For example, a bulk solid is not just a solid continuum since it takes the shape of the vessel containing it; it cannot be considered a liquid for it can be piled into heaps; and it is not a gas for it will not expand to fill the vessel containing it. Perhaps the material that flowing granular materials most resemble is that of a non-Newtonian fluid. Therefore, it seems reasonable to expect a theory for flowing granular materials to exhibit characteristics unique to viscoeclastic fluids such as the normal stress effects. For a thorough and up-to-date review of all these issues relevant to flows of granular materials we refer the reader to a recent article by Hutter and Rajagopal [5].

In a number of applications these materials are also heated prior to processing, or cooled after processing [6]. These contact dominated (dense phase) flows have applications in certain industrial equipment designed to heat, cool, or dry granular materials [7]. Sullivan and Sabersky [8] studied the heat transfer from a flat plate to various materials. They compared their experimental results to an idealized model which they called the "discrete particle model". Later, Spelt *et al.* [9] broadened the scope of the problem by studying the heat transfer to granular materials flowing along an inclined chute at higher velocities. Based on the experimental results, they speculated that the higher velocities caused a decrease in the density of the material and that the decrease in density caused a reduction in heat transfer. Gudhe *et al.* [10] studied the flow of granular materials down a heated inclined plane, where the effect of viscous dissipation was also considered.

Granular materials exhibit phenomena like normal stress differences in a simple shear flow, a phenomena usually referred to as *dilatancy* [11]. The normal-stress phenomenon is a characteristics of non-Newtonian fluids and non-linear elastic solids. One approach in the modeling of granular materials is to treat it as a *continuum*, which assumes that the material properties of the ensemble may be represented by continuous functions. One of the early continuum models for flowing granular materials based on the principles of modern continuum mechanics was proposed by Goodman and Cowin [12, 13]. This work was subsequently modified and improved upon by other investigators [14, 15]. Another approach used in the modeling of granular materials is based on the techniques used in the *kinetic theory* [16].

In general, and in many ways, flowing granular materials do behave like non-Newtonian fluids. The flow due to natural convection of some non-Newtonian fluids has received some attention in recent years. For example, Rajagopal and Na [17] studied the natural convection of a homogeneous incompressible fluid of grade three between two infinite parallel plates. They looked at the effect of the non-Newtonian nature of the fluid on the skin friction and the heat transfer coefficient. Szeri and Rajagopal [18] studied the flow of a third-grade fluid between two heated horizontal plates where the shear viscosity was assumed to be temperature dependent. Massoudi and Christie [19] studied the natural convection of a homogeneous incompressible fluid of grade three, between two infinite concentric vertical cylinders. Massoudi and Christie [20] studied the effects of variable viscosity and viscous dissipation on the flow of a third-grade fluid in a pipe.

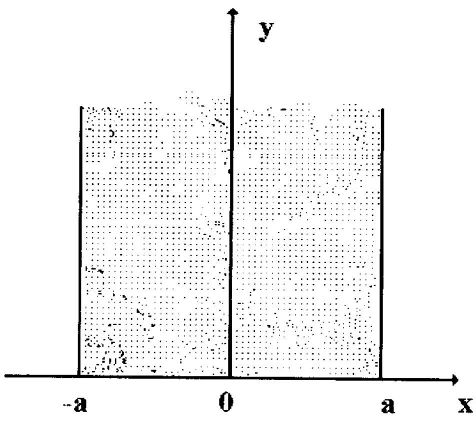

Fig. 1. A sketch of the theoretical model.

In this work we consider the natural convection of granular materials. The material is between two infinite vertical flat plates which are at two different temperatures (cf. Fig. 1). Using a continuum formulation for the stress tensor, the governing equations for the conservation of mass, momentum, and energy are developed. These equations are solved numerically using the numerical procedures developed by Phuoc and Durbetaki [21].

2. Governing equations

The basic governing equations for a continuum are the conservation of mass, momentum, energy, and the entropy inequality [22–24]. The conservation of mass takes the form

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{u}) = 0, \tag{1}$$

where $\partial/\partial t$ is the partial derivative with respect to time. The balance of linear momentum is

$$\rho \frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = \operatorname{div} \mathbf{T} + \rho \mathbf{b}, \tag{2}$$

where $\mathrm{d}/\mathrm{d}t$ is the material time derivative and $\mathbf{b}$ is the body force vector. The energy equation in its general form is

$$\rho \frac{\mathrm{d}\varepsilon}{\mathrm{d}t} = \mathbf{T} \cdot \mathbf{L} - \operatorname{div} \mathbf{q} + \rho r, \tag{3a}$$

where ε denotes the specific internal energy, $\mathbf{q}$ is the heat flux vector, r is the radiant heating, and $\mathbf{L}$ is the velocity gradient.

The Clausius–Duhem inequality is

$$\rho(\dot{\varepsilon} - \dot{\eta}\theta) \leqslant \mathbf{T} \cdot \mathbf{L} - (\mathbf{q} \cdot \operatorname{grad} \theta)/\theta, \tag{3b}$$

where η is the specific entropy and θ is the temperature and the dot denotes the material time derivative. For a complete study of a thermo-mechanical problem, the Second Law of Thermodynamics (the C–D inequality in this case), has to be included. In other words, in addition to other principles in continuum mechanics such as material symmetry, frame indifferent, etc., the Second Law also imposes certain restrictions on the type of the motion and/or on the constitutive parameters. Though there have been a few studies, including Goodman and Cowin [12, 13], which have proposed a thermodynamic analysis of a granular material, however, since there is no general agreement as to the behavior or the functional form of the constitutive relations and the fact that the Helmholtz free energy is not known, a complete treatment of the present model used in our study is lacking. Especially, since the later studies indicated that there were some discrepancies in the work of Goodman and Cowin [25]. We therefore will not attempt to carry out a thermodynamic analysis of the present model; rather we use the results of Rajagopal *et al.* [26, 27] to obtain either restrictions on the material parameters or to use a simple and reasonable expression for these undetermined parameters. In the next section we present the constitutive relations which are needed for the closure in the problem.

3. Constitutive relations

As we can see from Eqs. (2) and (3), we need to model the stress tensor $\mathbf{T}$ and the heat flux vector $\mathbf{q}$. Of course, the specific internal energy ε and the radiant heating r are also needed. However, due to the kinematical assumptions which we will make for the velocity and temperature fields, the left-hand side of Eq. (3a) will be automatically satisfied and therefore we do not need to specify ε. With regard to the radiation effect, we also assume that it is negligible, and therefore r is eliminated from our analysis. This leaves us with the task of modeling the stress tensor $\mathbf{T}$ and the heat flux vector $\mathbf{q}$. The stress tensor $\mathbf{T}$ for flowing granular materials has been the subject of extensive investigation in the last two decades. A detailed review of all the relevant issues can be found in Ref. [5]. In the present study, we will focus on a constitutive relation originally developed by Goodman and Cowin [12, 13] which has subsequently been modified.

We assume that the stress tensor $\mathbf{T}$ is given by [12, 28]

$$\mathbf{T} = \{\beta_0(\nu) + \beta_1(\nu)\nabla\nu \cdot \nabla\nu + \beta_2(\nu) \operatorname{tr} \mathbf{D}\}\mathbf{1} + \beta_4(\nu)\nabla\nu \otimes \nabla\nu + \beta_3(\nu)\mathbf{D}. \tag{4}$$

where ν denotes the volume fraction of the particles, $\mathbf{D}$ is the symmetric part of the velocity gradient, $\beta_0(\nu)$ is similar to pressure in a compressible fluid and is given by an equation of state, $\beta_2(\nu)$ is akin to the second coefficient of viscosity in a compressible fluid, $\beta_1(\nu)$ and $\beta_4(\nu)$ are the material parameters that reflect the distribution of the granular materials and $\beta_3(\nu)$ is the viscosity of the granular materials. The above model allows for normal-stress differences, a feature observed in granular materials. In general, the material properties β_0 through β_4 are functions of the density (or volume fraction ν), temperature, and the principle invariants of the stretching tensor $\mathbf{D}$, given by

$$\mathbf{D} = \tfrac{1}{2}[(\nabla \mathbf{u}) + (\nabla \mathbf{u})^{\mathrm{T}}], \tag{5}$$

where $\mathbf{u}$ is the velocity vector of the particles. In Eq. (4), $\mathbf{1}$ is the identity tensor, ∇ is the gradient operator, $\otimes$ indicates the outer (dyadic) product of two vectors, and tr designates the trace of a tensor. Furthermore, ν is related to the bulk density of the material ρ, through

$$\rho = \rho_s \nu \tag{6}$$

where ρ_s is the actual density of the grains at the place $\mathbf{x}$ and time t and the field ν is called the volume fraction (or the volume distribution) and is related to porosity n or the void ratio e by

$$\nu = 1 - n = \frac{1}{1+e}. \tag{7}$$

In general, the material parameters β_0 through β_4 have to be measured experimentally. Rajagopal *et al.* [27] have outlined such an experimental procedure in which an orthogonal rheometer can be used to provide useful information about some of these constitutive parameters.

With regard to the signs of β_0 through β_4, in Appendix A, we provide a sample calculation which indicates that $\beta_0(\nu)$ is similar to pressure in a compressible fluid and thus we can assume

$$\beta_0(\nu) = k\nu, \tag{8}$$

where

$$k < 0. \tag{9}$$

Rajagopal *et al.* [26] investigated the existence of solutions for the flow of a granular materials represented by Eq. (4), flowing down an inclined plane maintained at a constant temperature. They proved two existence theorems for a range of material parameters. Specifically, they found "*the existence of solutions in which ν decreases from the inclined plane to the free surface and vice versa. While the former is physically more intuitive with the densification of the material at the bottom due to the effect of gravity, the latter is also mathematically possible. We do not carry out any stability analysis of the solutions and it may be that the second solution can be ruled out on the basis of a stability analysis.*" The first theorem basically says that solutions exist for all values of $k < k_0$ where $k_0 < 0$ and for some values of $(\beta_1 + \beta_4)$. Existence was only proved for the cases when

$$\beta_1 + \beta_4 > 0. \tag{10}$$

Based on the above observation and the experimental evidence provided by Ahn [29] and Ahn *et al.* [30] where multiple solutions were also observed, we plan to do a parametric study. In accordance with the proposal made by Rajagopal and Massoudi [28] we use the following forms for β_1 through β_4, assuming that, in general, all these functions would decrease monotonically with ν. (The early works of Walton and Braun [31, 32] based on particle simulation indicated that the viscosity of flowing granular materials varied as a quadratic function of ν, while keeping $\mathbf{D}$ fixed.) Thus, assuming that material parameters β_0 through β_4 are smooth functions of ν, and using a Taylor's series expansion and neglecting terms of order ν^3 we have

$$\beta_1(\nu) = \beta_{10} + \beta_{11}\nu + \beta_{12}\nu^2, \tag{11}$$

$$\beta_2(\nu) = \beta_{20} + \beta_{21}\nu + \beta_{22}\nu^2, \tag{12}$$

$$\beta_3(\nu) = \beta_{30} + \beta_{31}\nu + \beta_{32}\nu^2, \tag{13}$$

$$\beta_4(\nu) = \beta_{40} + \beta_{41}\nu + \beta_{42}\nu^2, \tag{14}$$

where $\beta_{10} \ldots \beta_{42}$ are constants. Furthermore, as a limiting case, when there are no particles, i.e. $\nu = 0$, the stress tensor $\mathbf{T}$ should vanish. Therefore, the dependence of β_2 and β_3 on ν should be such that as $\nu \to 0$, these functions also vanish. Hence we must have

$$\beta_{20} = \beta_{30} = 0. \tag{15}$$

Johnson *et al.* [33, 34] also used the same representation for β_0 through β_4 as those proposed here.

Before we proceed with the discussion of the heat flux vector $\mathbf{q}$, we need to mention the pioneering work of Bagnold [35, 36] as they have a relevance to the present investigation. Bagnold [35] performed experiments on neutrally buoyant, spherical particles suspended in Newtonian fluids, undergoing shear in a coaxial rotating cylinder. He was able to measure the torque and the normal stress in the radial direction for various concentrations of the grains. He observed two distinct, limiting types of behavior. In the so-called "macro-viscous" region, which corresponds to low shear rates, the shear and normal stresses are linear functions of the velocity gradient. In this region, the fluid viscosity is the dominant parameter. In the second region, called the "grain-inertia region" the fluid in the interstices does not play an important role and the dominant effects arise from the particle–particle interactions. He also observed that both the shear and the normal stresses in this region are proportional to the square of the velocity gradient. This is basically a non-linear effect. The stress tensor which we use in this study captures the non-linear phenomenon due to normal stress differences. In addition, the kinematics of the present problem resemble in some way the "grain-inertia region" of Bagnold in that the interstitial effects due to fluid are ignored and the flow due to particle–particle interactions is due to natural convection.

Furthermore, we assume that the heat flux vector $\mathbf{q}$ satisfies Fourier's law, i.e.

$$\mathbf{q} = -K\nabla\theta, \tag{16}$$

where θ is the temperature and K is the thermal conductivity, which in general is a function of ν and θ.

4. Formulation of the problem

In general, whenever non-linear materials are concerned, the solution procedure, i.e. the numerical analysis for solving the governing equations becomes more complicated. As a result, exact solutions are indeed rare when it comes to the mechanics and heat transfer studies of non-linear materials or multiphase flows. Next to the exact solutions, finding solutions to simple limiting boundary value problems are extremely desirable. As it is, most of the constitutive relations used in mechanics, whether they are non-Newtonian fluids, turbulent models, etc., when substituted in the general governing or balance equations, would produce a system of partial differential equations which at times are impossible to solve completely or to find any solutions with the numerical techniques available. Therefore, from a modeling point of view, it is worthwhile to study problems where due to simplification of the kinematics of the flow, or the geometry of the case, or the boundary conditions used, we have a system of (non-linear) ordinary differential equations. The solution to these simpler problems would serve us, in at least two different ways: (i) providing us with some insight to the nature of these non-linear constitutive relations, and (ii) providing test cases where the solution to the general multidimensional equations can be compared to. With these in mind, for the problem under consideration, we make the following assumptions:

(i) the motion is steady;
(ii) radiant heating r can be ignored;
(iii) the constitutive equation for the stress tensor is given by Eq. (4) and the constitutive equation for the heat flux vector is that of Fourier's law, given by Eq. (16);
(iv) the density, velocity and temperature fields are of the form

$$\nu = \nu(x) \tag{17}$$

$$\mathbf{u} = u(x)\,\mathbf{j} \tag{18}$$

$$\theta = \theta(x) \tag{19}$$

$$\rho\mathbf{b} = -\rho_s\nu[1 - \gamma(\theta - \theta_m)]\,g\mathbf{j} \tag{20}$$

where θ_m is a reference temperature (for example $\theta_m = (\theta_1 + \theta_2)/2$), g is the acceleration due to gravity, and γ is the coefficient of thermal expansion;

(v) furthermore, we assume

$$\beta_0(v) = kv \tag{21}$$

$$\beta_3(v) = \beta_3^*(v + v^2), \tag{22}$$

where β_3^* is a constant and $\beta_1, \beta_2, \beta_4$ are assumed to be constants. Assuming that the Fourier's law of heat conduction is appropriate for the temperature range or conditions of this problem [37] we now need to specify the dependence of thermal conductivity on other parameters. There have been some studies with regard to porous materials [38–42], but few with regard to flowing granular materials [43]. In fact, Kaviany [39] presents a thorough review of the appropriate correlations for the effective thermal conductivity for packed beds (cf. Table 3.1, p. 129), and a discussion on the effective thermal conductivity in multiphase flows. In addition, Whitaker [44, 45] in his review articles also has provided insight into this issue. As a first step, in our analysis we will use a simple relationship where the thermal conductivity is a linear function of the volume fraction. [46, 47]

$$K = K_m(1 + 3\zeta v), \tag{23}$$

where $\zeta = (\psi - 1)/(\psi + 1)$. Here ψ is the ratio of conductivity of the particle to that of the matrix, and K_m is the thermal conductivity of the matrix.

We should remark here that due to the nature of the "limiting" approximations that we have made, Eqs. (17)–(19) represent kinematical assumptions and whether in reality there would ever be such a steady unidirectional flow is hard to say. But if such a solution to the approximated equations indeed exists, it would be interesting to know about it. Eq. (20) is basically the Boussinesq's approximation and it may not be a very good approximation in the present problem. There is no doubt that the interstitial fluid would play a role, and therefore, it is necessary to consider a better approximation than Eq. (20) in future. For a critical discussion and analysis of the Boussinesq's approximation, we refer the reader to Ref. [48]. Similarly, the limitations of Eqs. (21) and (22) are already discussed. This leaves us with the last phenomenological parameter in our problem, namely the thermal conductivity. As we mentioned before, there are many correlations, mostly based on the limited experimental data, where the dependence of the thermal conductivity on various other parameters such as temperature, volume fraction, etc. are depicted. We use Eq. (23) in the spirit of a simple approximation, indicating basically a linear dependence of the thermal conductivity on the volume fraction.

With the above assumptions, the conservation of mass is identically satisfied and from the balance of linear momentum and energy we get

$$k\frac{\mathrm{d}v}{\mathrm{d}x} + 2(\beta_1 + \beta_4)\frac{\mathrm{d}v}{\mathrm{d}x}\frac{\mathrm{d}^2v}{\mathrm{d}x^2} = 0, \tag{24}$$

$$\beta_3^*(v + v^2)\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + \beta_3^*(1 + 2v)\frac{\mathrm{d}v}{\mathrm{d}x}\frac{\mathrm{d}u}{\mathrm{d}x} = -\rho_s v[1 - \gamma(\theta - \theta_m)]g \tag{25}$$

$$2K_m(1 + 3\zeta v)\frac{\mathrm{d}^2\theta}{\mathrm{d}x^2} + 6K_m\zeta\frac{\mathrm{d}v}{\mathrm{d}x}\frac{\mathrm{d}\theta}{\mathrm{d}x} = -\beta_3^*(v + v^2)\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2. \tag{26}$$

The boundary conditions to be used in this problem are:

At $x = -a$:

$$u = 0, \tag{27a}$$

$$\theta = \theta_1. \tag{27b}$$

At $x = a$:

$$u = 0, \tag{28a}$$

$$\theta = \theta_2, \tag{28b}$$

where $\theta_1 > \theta_2$. For volume fraction, one can assume $v = v_1$ at $x = -a$ and $v = v_2$ at $x = a$. However, it should be noted that in general v_1 and v_2 are not known. Either these values are given based on experiments or one can do a parametric study. An alternative way of providing the necessary boundary conditions for v is to use the symmetry condition and mass-averaged quantity (see

[10]). The boundary conditions for v are assumed to be

$$\left(\frac{dv}{dx}\right)_{x=0} = 0, \tag{29}$$

$$Q = \int_{-a}^{a} v\, dx, \tag{30}$$

where Q is given. It is obvious from Eq. (24) that $dv/dx = 0$ or $v = \text{constant}$ is a solution to that equation.

5. Numerical solutions

Let us define the dimensionless distance ξ, the velocity Φ, and the temperature Γ by the following equation:

$$\xi = \frac{x}{a}; \quad \phi = \frac{u}{U_0}; \quad \Gamma = \frac{\theta - \theta_m}{\theta_1 - \theta_2}, \tag{31}$$

where U_0 is a reference velocity. The above (Eqs. (24)–(26)) become

$$\left(\frac{d^2 v}{d\xi^2} + R_1\right)\frac{dv}{d\xi} = 0, \tag{32}$$

$$R_2(v + v^2)\frac{d^2\phi}{d\xi^2} + R_2(1 + 2v)\frac{dv}{d\xi}\frac{d\phi}{d\xi} - v + R_4 v\Gamma = 0, \tag{33}$$

$$(1 + 3\zeta v)\frac{d^2\Gamma}{d\xi^2} + 3\zeta\frac{dv}{d\xi}\frac{d\Gamma}{d\xi} + R_3(v + v^2)\left[\frac{d\phi}{d\xi}\right]^2 = 0. \tag{34}$$

The boundary conditions become:

At $\xi = -1$:

$$\phi = 0; \quad \Gamma = \tfrac{1}{2} \tag{35}$$

At $\xi = 1$:

$$\phi = 0; \quad \Gamma = -\tfrac{1}{2} \tag{36}$$

and

$$\left(\frac{dv}{d\xi}\right)_{\xi=0} = 0, \tag{37}$$

$$N = \int_{-1}^{1} v\, d\xi, \tag{38}$$

where $N = Q/a$. The dimensionless parameters R_1, R_2, R_3, and R_4 are given by

$$R_1 = \frac{ka^2}{2(\beta_1 + \beta_4)}, \tag{39}$$

$$R_2 = \frac{\beta_3^* U_0}{2a^2 \rho_s g}, \tag{40}$$

$$R_3 = \frac{\beta_3^* U_0^2}{2K_m(\theta_1 - \theta_2)}, \tag{41}$$

$$R_4 = \gamma(\theta_1 - \theta_2). \tag{42}$$

These dimensionless parameters have the following physical meaning: R_1 is the ratio of the pressure force to the force developed in the material due to distribution of the void. R_2 is the ratio of the viscous force to the gravity force and R_3 is the product of the Prandtl number and the Eckert number. So in a sense, R_2 is related to a Reynolds number for the particles, R_3 is a measure of the importance of viscous dissipation, and R_4 is a measure of the buoyancy effects due to the thermal expansion.

Since Eq. (32) is independent of temperature and velocity, it can be solved separately. With the conditions given by Eqs. (37) and (38) the distribution of the volume fraction in the ξ-direction under the actions of the pressure force and the force developed in the materials is obtained as

$$v = -\frac{R_1\xi^2}{2} + \left(\frac{Q}{a} + \frac{R_1}{3}\right)\frac{1}{2}. \tag{43}$$

The momentum equation, Eq. (33), and the energy equation, Eq. (34), are coupled and must be solved together numerically using numerical procedures developed by Phuoc and Durbetaki [21] and Phuoc and Massoudi [49]. The technique uses the central difference approximation to discretize Eqs. (33) and (34). Since $d\phi/d\xi$ and $d\Gamma/d\xi$ are not known at $\xi = -1$, the technique requires that these unknowns must be initially guessed so that values of ϕ and Γ throughout the calculation domain can be evaluated. After the trial solutions are obtained the known boundary conditions of ϕ and Γ at

$\xi = 1$ are compared with the correspondent values provided by the trial solutions. If the solutions at this point do not agree with the known boundary conditions another guess must be used and the iteration is repeated. This procedure is continued until the solutions for ϕ and Γ provided by the initial guess conditions converge to the given values. In order to reduce the guess work, the Newton–Raphson method is used for correcting the initial guesses. The technique requires an additional set of $N_{ode} \times N_{unknown}$ (N_{ode} is the number of the governing equations and $N_{unknown}$ is the number of the unknown boundary conditions) differential equations that must be integrated simultaneously with the original governing equations. These equations, referred to as the perturbation equations, give the rate of change of the solutions of the original differential equations with respect to the guessed conditions. Thus, the technique reduces the guess work and the computer time and it is powerful for this type of application.

6. Results and discussions

Phuoc and Massoudi [49] have shown that when granular materials are packed between two vertical plates, which are at different temperatures, the materials near the hot plate are heated and the materials near the colder plate are cooled. As a result of such a temperature difference a natural convection flow ensues and the materials move. The motion of the materials is determined by parameters R_1–R_4 and it is always associated with a forward flow near the hot surface and a backward flow in the region near the colder surface. They report that the dimensionless parameter R_1 influences the particle distribution significantly and the velocity profiles to some extent but it does not have a significant effect either on the temperature distribution or the flow pattern of the materials. With regard to the effects of the dimensionless parameter R_3, they show that when R_3 is large the forward flow is dominant, the hot plate acts like a heat sink and the heat is transferred from the materials to the plate. When R_3 is small the reversed flow is significant and the hot plate is the heat source that heats the materials. The effects of parameter R_1 and R_3 on the velocity and temperature profiles are discussed in details by Phuoc and Massoudi [49] and will not be repeated here.

Figs. 2 and 3 demonstrate the velocity and temperature profiles respectively when R_1, R_3, R_4, and N are kept constant and R_2 is used as a parameter. The calculations are done for the case that the materials under consideration are uniformly distributed ($R_1 = 0$). Thus, the effect of $dv/d\xi$ is eliminated and the only effect that alters the velocity and temperature profiles is that of the parameter R_2. It is obvious that when R_2 increases from 0.1 to 100 the velocity profiles are significantly altered while only a slight change in the temperature profiles is obtained.

As defined above, the dimensionless parameter R_2 is the ratio of the viscous force to the gravity force. It is large when the viscous force is dominant and it is small when the gravity force is dominant. Thus, the effect of R_2 is that the natural convection flow of the materials might be in the gravity-controlled regime when R_2 is very small. If R_2 is increased the flow changes its characteristics and it shifts to the viscous-controlled regime. In general, however, it is believed that such a natural convention flow is in the gravity-viscous-controlled regime.

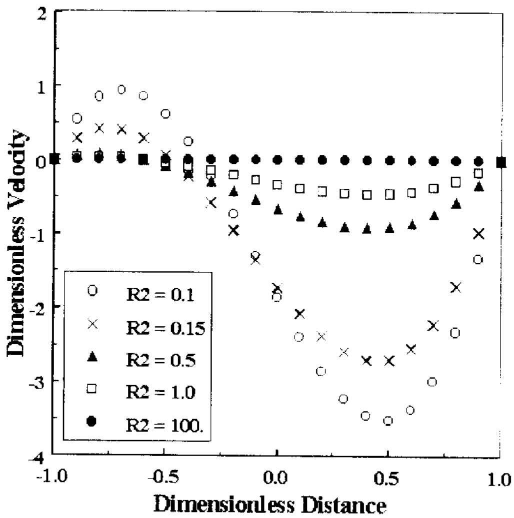

Fig. 2. Effect of the dimensionless parameter R_2 on the dimensionless velocity profiles ($N = 1$; $R_1 = 0$; $R_3 = 0.01$; $R_4 = 10$; $\zeta = 0.25$).

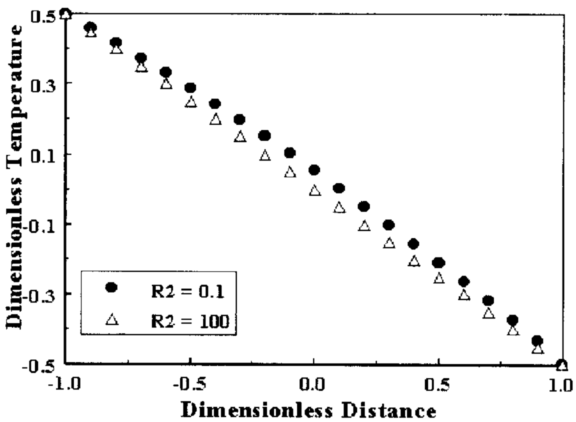

Fig. 3. Effect of the dimensionless parameter R_2 on the dimensionless temperature profiles ($N = 1$; $R_1 = 0$; $R_3 = 0.01$; $R_4 = 10$; $\zeta = 0.25$).

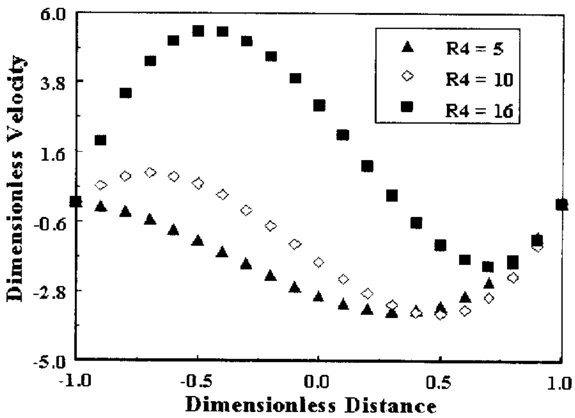

Fig. 4. Effect of the dimensionless parameter R_4 on the dimensionless velocity profiles ($N = 1$; $R_1 = 0$; $R_2 = 0.1$; $R_3 = 0.01$; $\zeta = 0.25$).

When the flow is in the gravity-controlled regime, the materials density, which depends on temperature, becomes an important factor that affects strongly the velocity profiles. A forward flow is expected to appear in the region where the materials are heated and a reversed flow is believed to occur in the region where the materials are cooled. Such a behavior is clearly shown in Fig. 2, for R_2 less than 100. For example, when $R_2 = 0.1$ the materials move strongly with the forward flow in the region near the hot plate and the reversed flow near the cold plate.

When the flow is in the viscous-controlled regime the flow becomes less intense and the materials might not move at all if R_2 is very large, even though the two hot and cold regions still exist. For example, from Fig. 2 it is clear that when $R_2 = 0.1$ both the forward and the reversed flows are easily detected. However, as R_2 increases to 0.5 or 1.0 the reversed flow remains visible while the forward flow is barely seen. When $R_2 = 100$, the velocity profile becomes almost flattened with the zerc velocity conditions at the two plates. Fig. 3 shows that changing R_2 does not have a significant effect on the temperature profiles.

To investigate the effects of the temperature difference between the two plates the calculations are done using R_4 as a parameter while other dimensionless parameters are kept constant. The results are shown in Figs. 4 and 5. Similar to R_3, parameter

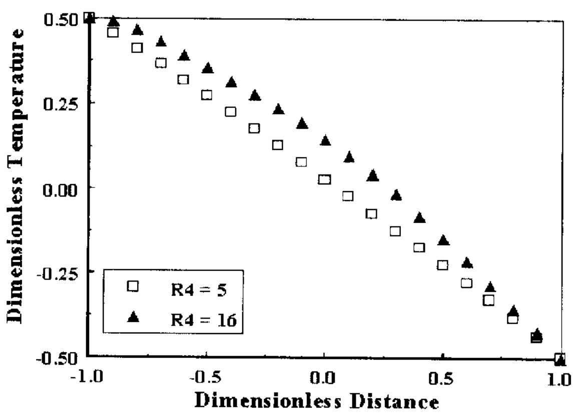

Fig. 5. Effect of the dimensionless parameter R_4 on the dimensionless temperature profiles ($N = 1$; $R_1 = 0$; $R_2 = 0.1$; $R_3 = 0.01$; $\zeta = 0.25$).

R_4 also shows a strong effect on both temperature and velocity of the materials between the two plates. That is, for small R_4, $(d\Gamma/d\xi)_{\xi-1}$ is negative and the material temperature is seen to decrease almost linearly from the hot plate to the cold plate. As R_4 increases, $(d\Gamma/d\xi)_{\xi-1}$ increases and becomes positive when R_4 becomes large. Thus, the temperature of the materials near the hot plate increases to higher than the temperature of the hot plate and then it decreases and reaches the cold plate temperature. For the velocity profile, the results indicate a significant forward flow when R_4 is large and

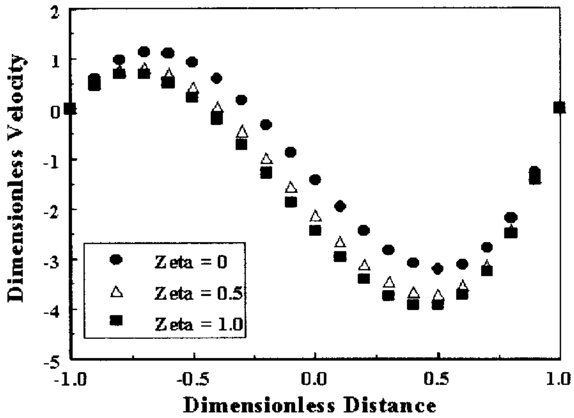

Fig. 6. Effect of the dimensionless parameter ζ on the dimensionless velocity profiles ($N = 1$; $R_1 = 0$; $R_2 = 0.1$; $R_3 = 0.01$; $R_4 = 10$).

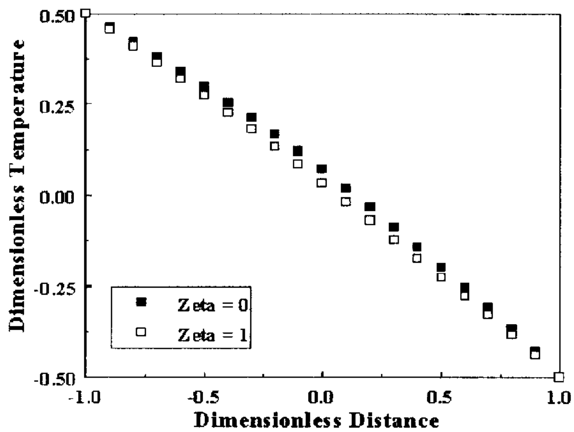

Fig. 7. Effect of the dimensionless parameter ζ on the dimensionless temperature profiles ($N = 1$; $R_1 = 0$; $R_2 = 0.1$; $R_3 = 0.01$; $R_4 = 10$).

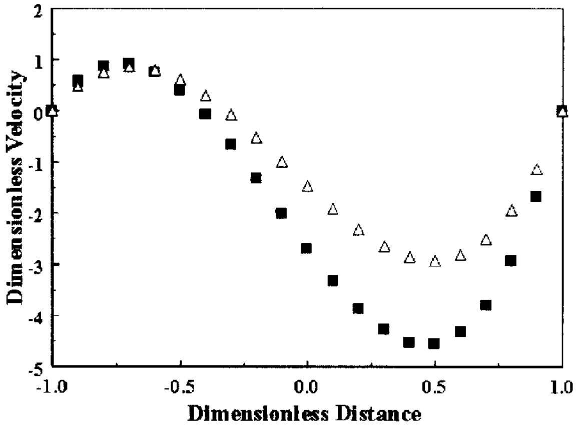

Fig. 8. Effect of the dimensionless parameter N on the dimensionless velocity profiles ($R_1 = 0$; $R_2 = 0.1$; $R_3 = 0.01$; $R_4 = 10$; $\zeta = 0.25$; ■: $N = 0.5$; △: $N = 1.5$).

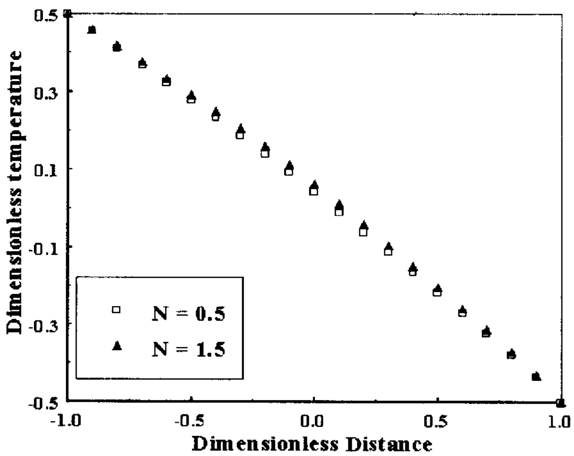

Fig. 9. Effect of the dimensionless parameter N on the dimensionless temperature profiles ($R_1 = 0$; $R_2 = 0.1$; $R_3 = 0.01$; $R_4 = 10$; $\zeta = 0.25$).

a significant reversed flow for smaller values of R_4. Since the forward flow is caused by the temperature difference that exists between the two plates, it must be significant when R_4 is large and it disappears when R_4 is small. This is shown in Fig. 4 where the forward flow is dominant when R_4 is greater than 10 and the reversed flow is the only flow that prevails when R_4 is less than 5.

The variation of the velocity and the temperature profiles with ζ as a parameter while others are kept constant are shown in Figs. 6 and 7. It is reminded that ζ is defined as the ratio of $(\psi-1)/(\psi+1)$. Here ψ is the ratio of conductivity of the particle to that of the matrix. Thus, meaningful values of ζ must be from 0 to 1. It is shown that ζ has little effect on both the velocity and the temperature profiles. The effect of the parameter N, which is the amount of material that is fed in, on the velocity and temperature profiles is shown in Figs. 8 and 9, respectively. While N has little effect on the temperature and the forward flow of the materials, it does affect the reversed flow significantly. Such an effect is due to the increase in the medium density which increases as the volume fraction of the materials increases.

7. Conclusions

The flow and heat transfer of an assembly of spherical particles, densely packed, between two vertical flat plates which are at different temperatures have been studied. A continuum model is used. For a fully developed flow of these materials, the governing equations reduce to a system of coupled, non-linear ordinary differential equations. The results show that the materials move with forward velocities near the hot plate and reversed velocities near the colder plate. The motion of the materials is determined by parameters R_1–R_4 which are defined to represent the effects of various transport properties and the competitions between the pressure force, the volume fraction distribution forces, and the gravity forces.

Nomenclature

$\mathbf{b}$	denotes the body force vector
$\mathbf{D}$	the symmetric part of the velocity gradient
e	void ratio e
g	acceleration due to gravity
K	thermal conductivity
K_m	thermal conductivity of the matrix.
$\mathbf{L}$	velocity gradient
n	porosity
Q	defined by Eq. (30)
$\mathbf{q}$	heat flux vector
r	radiant heating
R_1	defined by Eq. (39)
R_2	defined by Eq. (40)
R_3	defined by Eq. (41)
R_4	defined by Eq. (42)
$\mathbf{T}$	stress tensor
t	time
U_0	reference velocity
u	x-component velocity of the particles
$\mathbf{u}$	velocity vector of the particles
x	x-coordinate
β_0	similar to pressure in a compressible fluid
β_2	akin to the second coefficient of viscosity in a compressible fluid
β_1,β_4	material parameters that reflect the distribution of the granular materials
β_3	viscosity of the granular materials
β_3^*	constant
Γ	dimensionless temperature
γ	coefficient of thermal expansion
ε	specific internal energy
ζ	defined as $(\psi - 1)/(\psi + 1)$
θ_m	reference temperature $(=(\theta_1 + \theta_2)/2)$
θ	temperature
ν	volume fraction of the particles
ξ	dimensionless distance
ρ	actual density of the grains
ϕ	dimensionless velocity
ψ	ratio of conductivity of the particle to that of the matrix

Acknowledgment

The authors wish to thank Professor K. R. Rajagopal for his helpful suggestions and encouragements.

Appendix A

This is based on the report written by Rajagopal and Massoudi [28]. In order to obtain a possible representation for $\beta_0(\nu)$ in Eq. (8), we assume that the granular materials whose constitutive relation is given by Eq. (4) are in a large container and no flow is taking place (Static problem). Therefore

$$\mathbf{u} = 0, \tag{A1}$$

$$\mathbf{D} = 0. \tag{A2}$$

The equation of motion now reduces to

$$\operatorname{div} \mathbf{T} + \rho \mathbf{b} = 0 \tag{A3}$$

and the constitutive Eq. (4) becomes

$$\mathbf{T} = (\beta_0 + \beta_1 \nabla\nu \cdot \nu\nabla)\mathbf{1} + \beta_4 \nabla\nu \otimes \nabla\nu. \tag{A4}$$

Assuming that

$$v = v(y), \tag{A5}$$

where y is the positive upward direction, the y-component of Eq. (A3) becomes

$$\frac{\mathrm{d}}{\mathrm{d}y}\left[\beta_0(v) + \beta_1(v)\left(\frac{\mathrm{d}v}{\mathrm{d}y}\right)^2\right] + \frac{\mathrm{d}}{\mathrm{d}y}\left[\beta_4(v)\left(\frac{\mathrm{d}v}{\mathrm{d}y}\right)^2\right] - \rho_s v g = 0 \tag{A6}$$

where g is the acceleration due to gravity. Now, if we assume that, as a special case

$$\beta_1 = \beta_3 = 0, \tag{A7}$$

then the equilibrium Eq. (A6) is simplified to

$$\frac{\mathrm{d}}{\mathrm{d}y}[\beta_0(v)] = \rho_s v g. \tag{A8}$$

A Taylor series expansion for $\beta_0(v)$,

$$\beta_0(v) = \beta_{01} + \beta_0'(0)v + 0|v^2|, \tag{A9}$$

where $0|v^2|$ indicates terms of higher order than v. Now, if there are no particles, i.e. if $v = 0$, the stress tensor $\mathbf{T}$ should be zero. Substituting Eqs. (A9) and (A7) into Eq. (A4) and taking the limit as $v \to 0$, indicates that

$$\beta_{01} = 0. \tag{A10}$$

Therefore, Eq. (A9) becomes

$$\beta_0(v) = \beta_0'(0)v = kv, \tag{A11}$$

where k is a constant. Now, we can go back to the equilibrium Eq. (A8) and obtain the distribution of v. Therefore, substitutimg Eq. (A11) into (A8), gives

$$k\frac{\mathrm{d}v}{\mathrm{d}y} = \rho_s v g. \tag{A12}$$

This equation can easily be integrated and its solution is given by

$$v = A\mathrm{e}^{(\rho_s g/k)y}, \tag{A13}$$

where A is the constant of integration. Evaluating Eq. (A13) at two different heights y_1 and y_2, where $y_2 > y_1$, gives

$$v(y_2) = v(y_1)\,\mathrm{e}^{(\rho_s g/k)(y_2 - y_1)}. \tag{A14}$$

Physically, it is reasonable to expect, under regular conditions, that there would be more particles at the bottom of the container than at the top. This indicates that

$$v(y_1) > v(y_2). \tag{A15}$$

If this condition is to be met, we must have:

$$\frac{\rho_s g}{k} < 0. \tag{A16}$$

Since both ρ_s and g are positive, it follows that k must be negative ($k < 0$).

References

[1] K. Hutter, Theoretical Glaciology, Reidel, Kluwer Academic Publishers, Hingham, MA, 1983.

[2] M. Massoudi, E.J. Boyle, A review of theories for flowing granular materials with applications to fluidized beds and solids transport, DOE Topical Report, DOE/PETC/TR-91/8 (1991).

[3] S.L. Soo, Multiphase Fluid Dynamics, Science Press, Gower Technical, Brookfield 1990.

[4] K.R. Rajagopal, L. Tao, Mechanics of mixture, World Scientific, River Edge, NJ, 1995.

[5] K. Hutter, K.R. Rajagopal, On flows of granular materials, Contin. Mech. Thermodyn. 6 (1994) 81–139.

[6] J.S. Patton, R.H. Sabersky, C.E. Brennen, Convective heat transfer to rapidly flowing granular materials. Int. J. Heat Mass Transfer 29 (1986) 1263–1269.

[7] V.W. Uhl, W.L. Root, Heat transfer to granular solids in agitated units, Chem. Engng Prog. 63 (1967) 81–92.

[8] W.N. Sullivan, R.H. Sabersky, Heat transfer to flowing granular media, Int. J. Heat Mass Transfer 18 (1975) 97–107.

[9] J.K. Spelt, C.E. Brennen, S.H. Sabersky, Heat transfer to flowing granular materials, Int. J. Heat Mass Transfer 25 (1982) 791–796.

[10] R. Gudhe, K.R. Rajagopal, M. Massoudi, Fully developed flow of granular materials down a heated inclined plane, Acta Mech. 103 (1994) 63–78.

[11] O. Reynolds, Experiments showing dilatancy, a property of granular material, possibly connected with gravitation, Proc. Roy. Inst. Gr. Britain 11 (1886) 354–363.

[12] M.A. Goodman, S.C. Cowin, Two problems in the gravity flow of granular materials, J. Fluid Mech. 45 (1971) 312–339.

[13] M.A. Goodman, S.C. Cowin, A continuum theory for granular materials, Arch. Rational Mech. Anal. 44 (1972) 249–266.

[14] S.B. Savage, The mechanics of rapid granular flows, Adv. Appl. Mech. 24 (1984) 289–366.

[15] G. Ahmadi, A generalized continuum theory for granular materials, Int. J. Non-Linear Mech. 17 (1982) 21–33.
[16] E.J. Boyle, M. Massoudi, A theory for granular materials exhibiting normal stress effects based on Enkog's dense gas theory, Int. J. Engng Sci. 28 (1990) 1261–1275.
[17] K.R. Rajagopal, T.Y. Na, Natural convection flow of a non-Newtonian fluid between two vertical flat plates, Acta Mech. 54 (1985) 239–246.
[18] A.Z. Szeri, K.R. Rajagopal, Flow of a non-Newtonian fluid between heated parallel plates, Int. J. Non-Linear Mech. 20 (1985) 91–101.
[19] M. Massoudi, I. Christie, Natural convection flow of a non-Newtonian fluid between two concentric vertical cylinders, Acta Mech. 82 (1990) 11–19.
[20] M. Massoudi, I. Christie, Effects of variable viscosity and viscous dissipation on the flow of a third grade fluid in a pipe, Int. J. Non-Linear Mech. 30 (1995) 687.
[21] T.X. Phuoc, P. Durbetaki, Boundary layer analysis of unmixed combustible gases at the stagnation region: ignition and extinction. In: R.W. Lewis and K. Morgan (Eds.) Numerical Methods in Heat Transfer, Vol. 3 Wiley, New York, 1985 149.
[22] C. Truesdell, Rational Thermodynamics, 2nd Ed., Springer, New York, 1984.
[23] C. Truesdell, A First Course in Rational Continuum Mechanics. 2nd Ed. Academic Press Inc, San Diego (1991).
[24] C. Truesdell, W. Noll, The Non-Linear Field Theories of Mechanics, 2nd Ed., Springer, New York, 1992.
[25] S.B. Savage, Gravity flow of cohesionless granular materials in chutes and channels, J. Fluid Mech. 92 (1979) 53.
[26] K.R. Rajagopal, W. Troy, M. Massoudi, Existence of solutions to the equations governing the flow of granular materials, Eur. J. Mech., B Fluids, 11 (3) (1992) 265–276.
[27] K.R. Rajagopal, M. Massoudi, A.S. Wineman, Flow of granular materials between rotating disks, Mech. Res. Commun. 21 (6) (1994) 629–634.
[28] K.R. Rajagopal, M. Massoudi, A method for measuring material moduli of granular materials: flow in an orthogonal rheometer, Topical Report, U.S Department of Energy, DOE/PETC/TR-90/3, 1990.
[29] H. Ahn, Experimental, analytical investigation of granular materials, Ph.D. thesis, Caltech, 1989.
[30] H. Ahn, C.E. Brenner, R.H. Sabersky, Analysis of fully developed chute, flow of granular materials. ASME J. Appl. Mech 59 (1992) 109.
[31] O.R. Walton, R.L. Braun, Viscosity, granular temperature, and stress calculation for shearing assemblies of inelastic, frictional disks, J. Rheol. 30 (1986a) 949.
[32] O.R. Walton, R.L. Braun, Stress calculations for assemblies of inelastic spheres in uniform shear, Acta Mech. 63 (1986b) 73.
[33] G. Johnson, M. Massoudi, K.R. Rajagopal, Flow of a fluid–solid mixture between flat plates, Chem. Engng Sci. 46 (1991a) 1713.
[34] G. Johnson, M. Massoudi, K.R. Rajagopal, Flow of a fluid infused with solid particles through a pipe, Int. J. Engng Sci. 29 (1991b) 649.
[35] R.A. Bagnold, Experiments on a gravity-free dispersion of large solid spheres in a Newtonian fluid under shear, Proc. Roy. Soc. London, Series A, 225 (1954) 49.
[36] R.A. Bagnold, The shearing and dilatation of dry sand and the Singing mechanism, Proc. Royal. Soc. London, Ser. A 295 (1966) 219.
[37] Liu, I-Shih, On Fourier's law of heat conduction, Contin. Mech. Thermodyn. 2 (1990) 301.
[38] M. Kaviany, Principles of Covective Heat Transfer, Springer, New York, 1994.
[39] M. Kaviany, Principles of Heat Transfer in Porous Media, 2nd Ed. Springer, New York, 1995.
[40] D.A. Neild, A. Bejan, Convection in porous media. Springer, New York, 1992.
[41] Buyevich Yu.A., On the thermal conductivity of granular materials, Chem. Engng Sci. 24 (1974) 37.
[42] I.H. Tavman, Effective thermal conductivity of granular porous materials. Int. Common. Heat Mass Transfer 23 (1996) 169.
[43] M.L. Hunt, Comparison of convective heat transfer in packed beds and granular flows. Annu. Rev. Heat Transfer 3 (1990) 163.
[44] S. Whitaker, Simultaneous heat, mass, and momentum transfer in porous media: A theory of drying, Adv. Heat Transfer 13 (1977) 119.
[45] S. Whitaker, Heat and mass transfer in granular porous media. In: A.S. Majumda (Ed.) Advances in Drying, Hemisphere, New York, (1981) 1–46.
[46] Y.M. Bashir, J.D, Goddard, Experiments on the conductivity of suspensions of ionically conductive spheres, AIChE J. 36 (1990) 387–396.
[47] G.K. Batchelor, R.W. OBrien, Thermal or electrical conduction through a grannular material, Proc. Roy. Soc. London, Ser. A 355 (1977) 313–333.
[48] R.N. Hills, P.H. Roberts, On the motion of a fluid that is incompressible in a generalized sense and its relationship to the Boussinesq approximation, Stability Appl. Anal. Contin. Media 1 (1991) 205.
[49] T.X. Phuoc, M. Massoudi, A numerical study of the flow of granular materials between two vertical flat plates which are at different temperatures, submitted for publication.
[50] J.E. Dunn, R.L. Fosdick, Thermodynamics, stability and boundedness of fluids of complexity 2 and fluids of second grade, Arch. Rational Mech. Anal 56 (1974) 191.
[51] J.E. Dunn, K.R. Rajagopal, Fluids of different types: critical review and thermodynamic analysis, Int. J. Engng Sci. 33 (1995) 689.
[52] R.L. Fosdick, K.R. Rajagopal, Thermodynamics and Stability of fluids of third grade, Proc. Roy. Soc. London A. 339 (1980) 351.
[53] I.A. Müller, A thermodynamic theory of mixtures of fluids, Arch. Rational Mech. Anal. 28 (1968) 1.